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Fixed points in Peano arithmetic with ordinals

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Abstract

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This paper deals with some proof-theoretic aspects of fixed point theories over Peano arithmetic with ordinals. It studies three such theories which differ in the principles which are available for induction on the natural numbers and ordinals. The main result states that there is a natural theory in this framework which is a conservative extension of Peano arithmetic.

1. Introduction

Several publications in recent years have presented various formal theories in which fixed points of so-called inductive operator forms can be represented. Starting point for these proof-theoretic activities is the famous first-order theory ID_1 of (noniterated) inductive definitions, cf. e.g. Buchholz et al. [1] and Feferman [3]. It is formulated in the language $L_1(FP)$ which extends the usual language L_1 of first-order arithmetic by adding fixed point constants $\mathcal{P}_A$ for all P-positive L_1 formulas $A(P, x)$. The axioms of ID_1 comprise the axioms of Peano arithmetic PA plus

$$\text{(I)} \quad (\forall x)[A(\mathcal{P}_A, x) \to \mathcal{P}_A(x)],$$

$$\text{(II)} \quad (\forall x)[A(\varphi, x) \to \varphi(x)] \to (\forall x)[\mathcal{P}_A(x) \to \varphi(x)]$$

for all constants $\mathcal{P}_A$ and formulas $\varphi(x)$ of $L_1(FP)$. Hence ID_1 formalizes that $\mathcal{P}_A$ represents the least definable fixed point of (the operator associated to) $A(P, x)$. The theory ID_1 is already fairly strong and goes beyond the so-called predictive mathematics.

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Later Aczel, Feferman and Friedman introduced predictive subsystems of ID_1, in particular the theory $\widehat{ID}_1$, cf. e.g. Feferman [5]. In this system the axioms (I) and (II) are replaced by fixed point axioms

$$(\forall x)[A(\mathcal{P}_A, x) \leftrightarrow \mathcal{P}_A(x)]$$

for all constants $\mathcal{P}_A$. Now $\mathcal{P}_A$ represents a fixed point of $A(P, x)$, but not necessarily its least fixed point. The transition from least fixed points to fixed points has the effect that $\widehat{ID}_1$ is a predictive theory which is proof-theoretically equivalent to the subsystem $(\Sigma^1_1\text{-}AC)$ of analysis.

At lot is known about the proof-theoretic aspects of theories for inductive definitions and fixed point theories, and often the proof-theoretic methods exploit concepts which go back to the definability theory of inductive definitions, presented for example in Moschovakis [12]. One of the basic concepts there is the stratification of the least fixed point I_A of an inductive operator form $A(P, x)$ into stages. One simply defines by recursion on the ordinals

$$I_A^\alpha := \left\{n : A\Big(\bigcup_{\beta<\alpha} I_A^\beta, n\Big)\right\}$$

and obtains that I_A is the union of its stages I_A^α. Powerful frameworks for discussing the connections between proof theory and definability theory and for exploiting these connections for proof-theoretic studies are infinitary systems and theories for admissible sets; cf. Jäger [8–11] and Pohlers [13–16] for further information and references.

In this paper we introduce another natural framework for a proof-theoretic approach to inductive definitions and fixed point theories which is in a certain sense minimal for simulating the definability-theoretic approach to inductive definitions. We simply add to L_1 a new sort of variables, so-called ordinal variables, a binary relation symbol $<$ for the less relation on the ordinals and a binary relation symbol P_A for each inductive operator form $A(P, x)$. Then we present three theories, PA^r_Ω, PA^w_Ω and PA_Ω, which extend PA by several axioms (see p. 123) to ensure that the predicates $P_A(\alpha, x)$ behave like the stages of the corresponding inductive definition and that the predicate $P_A(x) := (\exists\alpha)P_A(\alpha, x)$ represents a fixed point of $A(P, x)$. These axioms include the *inductive operator axioms*

$$P_A^\alpha(s) \leftrightarrow A(P_A^{<\alpha}, s)$$

and axioms for Σ *reflection on the ordinals,* which correspond to the Σ reflection axioms of Kripke–Platek set theory,

$$\varphi \rightarrow (\exists\alpha)\varphi^\alpha$$

for all formulas without positive unbounded universal quantifiers and negative unbounded existential quantifiers[1].

[1] φ^α denotes the formula which is obtained by replacing all unbounded ordinal quantifiers $(Q\beta)$ in φ by $(Q\beta < \alpha)$.

PA^r_Ω, PA^w_Ω and PA_Ω differ in their induction principles: In PA^r_Ω induction on the natural numbers and ordinals is restricted to Δ^Ω_0 formulas, i.e., formulas which do not contain unbounded ordinal quantifiers. In PA^w_Ω we have the schema of induction on the natural numbers for all formulas and Δ^Ω_0 induction on the ordinals. PA_Ω, finally, comprises the schemas of induction on the natural numbers and ordinals for all formulas.

It follows from earlier work (cf. [8, 9]) that PA_Ω is proof-theoretically equivalent to ID_1 and PA^w_Ω to $\widehat{ID}_1$. The main technical work of this paper, presented in Section 3, is to show that PA^r_Ω is a conservative extension of PA. This result is interesting since it shows that it is possible to have a natural fixed point theory of small proof-theoretic strength which, on the other hand, is close to the traditional approach in definability theory. Our results about PA^r_Ω and PA^w_Ω will also be used for the proof-theoretic analysis of systems of explicit mathematics with non-constructive μ-operator in Feferman and Jäger [6].

There is related work which should be mentioned. Feferman [4] deals with a theory T_Ω of operations, classes and ordinals. Instead of Peano arithmetic PA, the system T_Ω is based on (a fragment of) Feferman's explicit mathematics. T_Ω was developed for model-theoretic purposes, and a proof-theoretic analysis of T_Ω is not given in [4]. However, it seems that T_Ω is of about the same proof-theoretic strength as PA_Ω. Cantini [2] is concerned with theories of partial classifications and a theory of Frege structures extended by the Myhill–Flagg hierarchy of implications. In this enterprise he also deals with theories which have some connections to PA^r_Ω and obtains results similar to our proof-theoretic analysis of PA^r_Ω. However, the following approach is more direct and uniform.

2. The theories PA^r_Ω, PA^w_Ω and PA_Ω

Let L_1 be any of the usual first-order languages of arithmetic with number variables $x, y, z, \ldots$ (possibly with subscripts), the constant 0, as well as function and relation symbols for all primitive recursive functions and relations. $L_1(P)$ then is the extension of L_1 by a new n-ary relation symbol P. An $L_1(P)$ formula is called *P-positive* if each occurrence of P in this formula is positive. We call P-positive formulas which contain at most $x_1, \ldots, x_n$ free n-ary *inductive operator forms*, and let $A(P, x_1, \ldots, x_n)$ range over such forms. In the following we restrict ourselves to unary inductive operator forms in order to keep the notation as simple as possible. It is obvious, however, how all definitions, theorems and arguments can be generalized to arbitrary n.

Now we extend L_1 to a new first-order language L_Ω by adding a new sort of *ordinal variables* $\alpha, \beta, \gamma, \ldots$ (possibly with subscripts), a new binary relation symbol $<$ for the less relation on the ordinals[2] and a binary relation symbol P_A for each inductive operator form $A(P, x)$.

[2] To avoid confusion we use $<_N$ as symbol for the primitive recursive less relation on the nonnegative integers.

The *number terms* $s, t, \ldots$ (possibly with subscripts) of L_Ω are the number terms of L_1; the *ordinal terms* of L_Ω are the ordinal variables. The *formulas* $\varphi, \psi, \chi, \theta, \ldots$ (possibly with subscripts) of L_Ω are inductively generated as follows:

1. If R is an n-ary relation symbol of L_1, then $R(s_1, \ldots, s_n)$ is an (atomic) formula of L_Ω.
2. $(\alpha < \beta)$, $(\alpha = \beta)$ and $P_A(\alpha, s)$ are (atomic) formulas of L_Ω. We write $P_A^\alpha(s)$ for $P_A(\alpha, s)$.
3. If φ and ψ are formulas of L_Ω, then $\neg\varphi$, $(\varphi \vee \psi)$, $(\varphi \wedge \psi)$ and $(\varphi \rightarrow \psi)$ are formulas of L_Ω.
4. If φ is a formula of L_Ω, then $(\exists x)\varphi$ and $(\forall x)\varphi$ are formulas of L_Ω.
5. If φ is a formula of L_Ω, then $(\exists \alpha)\varphi$ and $(\forall \alpha)\varphi$ are formulas of L_Ω.
6. If φ is a formula of L_Ω, then $(\exists \alpha < \beta)\varphi$ and $(\forall \alpha < \beta)\varphi$ are formulas of L_Ω.

Parentheses can be omitted if there is no danger of confusion. The boldface notation $\boldsymbol{v}$ is used as a shorthand for a finite string $v_1, \ldots, v_n$ of expressions whose length will be specified by the context. We write $\varphi[\boldsymbol{\alpha}]$ to indicate that all free ordinal variables of the formula φ come from the list $\boldsymbol{\alpha}$; $\varphi(\boldsymbol{\alpha})$ may contain other free ordinal variables besides $\boldsymbol{\alpha}$. Both $\varphi[\boldsymbol{\alpha}]$ and $\varphi(\boldsymbol{\alpha})$, may contain free number variables. If $\varphi(P)$ is an $L_1(P)$ formula and $\psi(x)$ an L_Ω formula, then $\varphi(\psi)$ denotes the result of substituting $\psi(s)$ for every occurrence of $P(s)$ in $\varphi(P)$. For every L_Ω formula φ we write φ^α to denote the L_Ω formula which is obtained by replacing all unbounded quantifiers $(Q\beta)$ in φ by $(Q\beta < \alpha)$. Additional abbreviations are:

$$(\varphi \leftrightarrow \psi) := ((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)),$$
$$P_A^{<\alpha}(s) := (\exists \beta < \alpha) P_A^\beta(s),$$
$$P_A(s) := (\exists \alpha) P_A^\alpha(s).$$

The following inductively defined subclasses of the L_Ω formulas will be important for the definition of the theories PA_Ω^r, PA_Ω^w and PA_Ω.

Definition 1 (*Δ_0^Ω formulas*)
1. Every atomic formula of L_Ω is a Δ_0^Ω formula.
2. If φ and ψ are Δ_0^Ω formulas, then $\neg\varphi$, $(\varphi \vee \psi)$, $(\varphi \wedge \psi)$ and $(\varphi \rightarrow \psi)$ are Δ_0^Ω formulas.
3. If φ is a Δ_0^Ω formula, then $(\exists x)\varphi$ and $(\forall x)\varphi$ are Δ_0^Ω formulas.
4. If φ is a Δ_0^Ω formula, then $(\exists \alpha < \beta)\varphi$ and $(\forall \alpha < \beta)\varphi$ are Δ_0^Ω formulas.

Definition 2 (*Σ^Ω and Π^Ω formulas*)
1. Every Δ_0^Ω formula is a Σ^Ω and Π^Ω formula.
2. If φ is a Σ^Ω formula (Π^Ω formula), then $\neg\varphi$ is a Π^Ω formula (Σ^Ω formula).
3. If φ and ψ are Σ^Ω formulas (Π^Ω formulas), then $(\varphi \vee \psi)$ and $(\varphi \wedge \psi)$ are Σ^Ω formulas (Π^Ω formulas).

4. If φ is a Π^{Ω} formula (Σ^{Ω} formula) and ψ a Σ^{Ω} formula (Π^{Ω} formula), then $(\varphi \rightarrow \psi)$ is a Σ^{Ω} formula (Π^{Ω} formula).
5. If φ is a Σ^{Ω} formula (Π^{Ω} formula), then $(\exists x)\varphi$ and $(\forall x)\varphi$ are Σ^{Ω} formulas (Π^{Ω} formulas).
6. If φ is a Σ^{Ω} formula (Π^{Ω} formula), then $(\exists \alpha < \beta)\varphi$ and $(\forall \alpha < \beta)\varphi$ are Σ^{Ω} formulas (Π^{Ω} formulas).
7. If φ is a Σ^{Ω} formula, then $(\exists \alpha)\varphi$ is a Σ^{Ω} formula.
8. If φ is a Π^{Ω} formula, then $(\forall \alpha)\varphi$ is a Π^{Ω} formula.

The collection of all Σ^{Ω} and Π^{Ω} formulas is denoted by $\mathcal{U}^{\Omega}$ so that $\varphi \in \mathcal{U}^{\Omega}$ if and only if $\neg\varphi \in \mathcal{U}^{\Omega}$. It is also clear that all L_1 formulas are Δ_0^{Ω} formulas.

The complexity of L_{Ω} formulas is measured by their rank $rn(\varphi)$ which is inductively defined as follows.

1. If φ belongs to $\mathcal{U}^{\Omega}$, then $rn(\varphi) := 0$.
2. For L_{Ω} formulas not in $\mathcal{U}^{\Omega}$ we define their rank according to the following rules:

$$rn(\neg\psi) := rn(\psi) + 1,$$
$$rn(\psi j \chi) := \max(rn(\psi), rn(\chi)) + 1,$$
$$rn((Qx)\psi) := rn(\psi) + 1,$$
$$rn((Q\alpha)\psi) := rn(\psi) + 1,$$
$$rn((Q\alpha < \beta)\psi) := rn(\psi) + 2.$$

By an L_{Ω} *theory* we mean a (possibly infinite) collection of L_{Ω} formulas, and we write $T \vdash \varphi$ to express that the L_{Ω} formula φ can be derived from the L_{Ω} theory T by the usual axioms and rules of predicate logic with equality in both sorts.

Now we introduce three L_{Ω} theories which differ in the strength of their induction principles. The weakest of those, PA_{Ω}^{r}, is given by the following axioms:

Number-theoretic axioms. These comprise the axioms of Peano arithmetic PA with exception of complete induction on the natural numbers.

Inductive operator axioms. For all inductive operator forms $A(P, x)$:

$$P_A^{\alpha}(s) \leftrightarrow A(P_A^{<\alpha}, s).$$

Σ^{Ω} *reflection axioms.* For every Σ^{Ω} formula φ:

(Σ^{Ω}-Ref) $\quad \varphi \rightarrow (\exists \alpha)\varphi^{\alpha}$.

Linearity of the relation $<$ *on the ordinals.*

(LO) $\quad \alpha \not< \alpha \wedge (\alpha < \beta \wedge \beta < \gamma \rightarrow \alpha < \gamma) \wedge (\alpha < \beta \vee \alpha = \beta \vee \beta < \alpha)$.

Δ_0^Ω *induction on the natural numbers.* For all Δ_0^Ω formulas $\varphi(x)$:

$(\Delta_0^\Omega\text{-IND}_N)$ $\quad \varphi(0) \wedge (\forall x)[\varphi(x) \rightarrow \varphi(x')] \rightarrow (\forall x)\varphi(x)$.

Δ_0^Ω *induction on the ordinals.* For all Δ_0^Ω formulas $\varphi(\alpha)$:

$(\Delta_0^\Omega\text{-IND}_\Omega)$ $\quad (\forall \alpha)[(\forall \beta < \alpha)\varphi(\beta) \rightarrow \varphi(\alpha)] \rightarrow (\forall \alpha)\varphi(\alpha)$.

PA_Ω^w is the extension of PA_Ω^r by the following *scheme of complete induction on the natural numbers*:

$(L_\Omega\text{-IND}_N)$ $\quad \varphi(0) \wedge (\forall x)[\varphi(x) \rightarrow \varphi(x')] \rightarrow (\forall x)\varphi(x)$

for all L_Ω formulas $\varphi(x)$. PA_Ω is the extension of PA_Ω^w by the following *scheme of induction on the ordinals*

$(L_\Omega\text{-}IND_\Omega)$ $\quad (\forall \alpha)[(\forall \beta < \alpha)\varphi(\beta) \rightarrow \varphi(\alpha)] \rightarrow (\forall \alpha)\varphi(\alpha)$.

for all L_Ω formulas $\varphi(\alpha)$. There are interesting theories between PA_Ω^r and PA_Ω^w as well as between PA_Ω^w and PA_Ω which are obtained by restricting $(L_\Omega\text{-IND}_N)$ and $(L_\Omega\text{-IND}_\Omega)$ to various classes of formulas (e.g. Σ^Ω formulas). But in this article we will not go into this direction and confine ourselves to PA_Ω^r, PA_Ω^w and PA_Ω.

From the inductive operator and Σ^Ω reflection axioms we can easily deduce that the Σ^Ω formula P_A describes a fixed point of the inductive operator form $A(P, x)$. If $(L_\Omega\text{-IND}_\Omega)$ is available as well, then this fixed point can be proved to be the least L_Ω definable fixed point of $A(P, x)$. These constitute the following statement, whose proof is left to the reader.

Theorem 3. *We have for all operator forms $A(P, x)$ of $L_1(P)$ and all formulas $\varphi(x)$ of L_Ω:*

(1) $PA_\Omega^r \vdash (\forall x)[P_A(x) \leftrightarrow A(P_A, x)]$,

(2) $PA_\Omega \vdash (\forall x)[A(\varphi, x) \rightarrow \varphi(x)] \rightarrow (\forall x)[P_A(x) \rightarrow \varphi(x)]$.

This theorem suggests that there is a close relationship between the theory PA_Ω and the well known theory ID_1 (cf. e.g. [1, 3]) as well as between PA_Ω^w and the fixed point theory $\widehat{ID}_1$ of Feferman [5]. Both theories, ID_1 and $\widehat{ID}_1$, are formulated in the language $L_1(FP)$ which extends L_1 by adding fixed point constants $\mathscr{P}_A$ for all inductive operator forms $A(P, x)$. As mentioned earlier, ID_1 has as axioms the axioms of Peano arithmetic PA with the scheme of complete induction on the natural numbers for all formulas of the language $L_1(FP)$ plus

(I) $\quad (\forall x)[A(\mathscr{P}_A, x) \rightarrow \mathscr{P}_A(x)]$,

(II) $\quad (\forall x)[A(\varphi, x) \rightarrow \varphi(x)] \rightarrow (\forall x)[\mathscr{P}_A(x) \rightarrow \varphi(x)]$

for all constants $\mathscr{P}_A$ and formulas $\varphi(x)$ of $L_1(FP)$. The theory $\widehat{ID}_1$ is the subsystem of ID_1 wtih (I) and (II) replaced for all constants $\mathscr{P}_A$ by the fixed point axioms

$$(\forall x)[A(\mathcal{P}_A, x) \leftrightarrow \mathcal{P}_A(x)].$$

There is a natural translation of $L_1(FP)$ into L_Ω: One only has to interpret the atomic formulas $\mathcal{P}_A(x)$ of $L_1(FP)$ by the Σ^Ω formulas $P_A(x)$ of L_Ω. Hence complete induction on the natural numbers for L_1 formulas is a consequence of (Δ_0^Ω-IND$_N$), whereas (L_Ω-IND$_N$) is needed to prove the translations of complete induction on the natural numbers for $L_1(FP)$ formulas.

Obviously PA^r_Ω contains PA. Although the (translations of the) fixed point axioms of $\widehat{ID}_1$ are provable in PA^r_Ω according to the previous theorem, we need (L_Ω-IND$_N$) for dealing with the scheme of complete induction which is available in $\widehat{ID}_1$ for all $L_1(FP)$ formulas. Hence $\widehat{ID}_1$ can be directly interpreted in PA^w_Ω but not in PA^r_Ω. Finally we also obtain from Theorem 3 that PA_Ω contains ID_1.

The results and techniques of Jäger [8, 9] establish the exact relationship of PA_Ω and PA^w_Ω to suitable systems of Kripke–Platek set theory with the natural numbers as urelements and yield that they are conservative extensions of ID_1 and $\widehat{ID}_1$, respectively[3]. Actually, the following theorem can be generalized even to all formulas of $L_1(FP)$ which do not contain negative occurrences of the fixed point constants.

Theorem 4. (1) *PA_Ω is a conservative extension of ID_1 with respect to all L_1 formulas.*

(2) *PA^w_Ω is a conservative extension of $\widehat{ID}_1$ with respect to all L_1 formulas.*

However, the results of [8, 9] do not provide a proof-theoretic treatment of PA^r_Ω. This will be achieved in the following section.

3. Proof-theoretic strength of PA^r_Ω

It is the aim of this section to show that PA^r_Ω is a conservative extension of Peano arithmetic PA. To this end we introduce an auxiliary system G_Ω which is a Gentzen style reformulation of PA^r_Ω. The capital Greek letters Γ, Λ, $\Phi, \ldots$ (possibly with subscripts) denote finite sequences of L_Ω formulas, and *sequents* are formal expressions of the form $\Gamma \supset \Lambda$. We write $\Gamma[\boldsymbol{\alpha}] \supset \Lambda[\boldsymbol{\alpha}]$ to express that all formulas in Γ and Λ are of the form $\varphi[\boldsymbol{\alpha}]$.

The system G_Ω is an extension of the classical Gentzen calculus LK (cf. e.g. Girard [7] and Takeuti [18]) by additional $\mathcal{U}^\Omega$ axioms and rules for bounded quantification, operators, Σ reflection and Δ_0^Ω induction. The axioms and rules of G_Ω can be divided into the following seven classes.

[3] It is also possible to obtain these results by a direct proof-theoretic analysis of PA_Ω and PA^w_Ω without making use of systems of Kripke–Platek set theory.

Logical and $\mathcal{U}^\Omega$ axioms. For all Δ_0^Ω formulas φ and all axioms ψ of PA^r_Ω which are $\mathcal{U}^\Omega$ formulas:

$$\varphi \supset \varphi, \qquad \supset \psi.$$

Structural rules. The structural rules consist of the usual weakening, exchange and contraction rules.

Propositional rules. The propositional rules consist of the usual rules for introducing the propositional connectives on the left- and right-hand sides of sequents.

Quantifier rules. Formulated for existential quantifiers; the corresponding rules for universal quantifiers must also be included. By $(*)$ we mark the rules where the designated free variables are not to occur in the conclusion.

$$\frac{\Gamma \supset \Lambda,\ \varphi(s)}{\Gamma \supset \Lambda,\ (\exists x)\varphi(x)}, \qquad \frac{\Gamma,\ \varphi(y) \supset \Lambda}{\Gamma,\ (\exists x)\varphi(x) \supset \Lambda}(*),$$

$$\frac{\Gamma \supset \Lambda,\ \varphi(\gamma)}{\Gamma \supset \Lambda,\ (\exists \alpha)\varphi(\alpha)}, \qquad \frac{\Gamma,\ \varphi(\gamma) \supset \Lambda}{\Gamma,\ (\exists \alpha)\varphi(\alpha) \supset \Lambda}(*),$$

$$\frac{\Gamma \supset \Lambda,\ \gamma < \beta \wedge \varphi(\gamma)}{\Gamma \supset \Lambda,\ (\exists \alpha < \beta)\varphi(\alpha)}, \qquad \frac{\Gamma,\ \gamma < \beta \wedge \varphi(\gamma) \supset \Lambda}{\Gamma,\ (\exists \alpha < \beta)\varphi(\alpha) \supset \Lambda}(*).$$

Rule for Σ reflection. For all Σ^Ω formulas φ and ordinal variables α which are not free in φ:

$$\frac{\Gamma \supset \Lambda,\ \varphi}{\Gamma \supset \Lambda,\ (\exists \alpha)\varphi^\alpha}.$$

Rule for Δ_0^Ω induction on the ordinals. For all Δ_0^Ω formulas $\varphi(\alpha)$:

$$\frac{\Gamma \supset \Lambda,\ (\forall \alpha)[(\forall \beta < \alpha)\varphi(\beta) \rightarrow \varphi(\alpha)]}{\Gamma \supset \Lambda,\ (\forall \alpha)\varphi(\alpha)}.$$

Cut rule.

$$\frac{\Gamma \supset \Lambda,\ \varphi \qquad \Gamma,\ \varphi \supset \Lambda}{\Gamma \supset \Lambda}.$$

The notion $G_\Omega \vdash^n_r \Gamma \supset \Lambda$ is used to express that the sequent $\Gamma \supset \Lambda$ is provable in G_Ω by a proof of length n so that all cut formulas have rank less than r; it is inductively defined as follows:

1. If $\Gamma \supset \Lambda$ is an axiom of G_Ω, then we have $G_\Omega \vdash^n_r \Gamma \supset \Lambda$ for all natural numbers n and r.

2. If $G_\Omega \vdash_r^{n_\iota} \Gamma_\iota \supset \Lambda_\iota$ and $n_\iota < n$ for every premise $\Gamma_\iota \supset \Lambda_\iota$ of a G_Ω rule which is not a cut, then we have $G_\Omega \vdash_r^n \Gamma \supset \Lambda$ for the conclusion $\Gamma \supset \Lambda$ of this rule.

3. If $G_\Omega \vdash_r^{n_\iota} \Gamma_\iota \supset \Lambda_\iota$ and $n_\iota < n$ for the two premises $\Gamma_\iota \supset \Lambda_\iota$ of a cut rule ($\iota = 1, 2$) with cut formula φ so that $rn(\varphi) < r$, then we have $G_\Omega \vdash_r^n \Gamma \supset \Lambda$ for the conclusion $\Gamma \supset \Lambda$ of this cut.

Hence the sequent $\Gamma \supset \Lambda$ is cut-free provable in G_Ω if there exists a natural number n with $G_\Omega \vdash_0^n \Gamma \supset \Lambda$. On the other hand, $G_\Omega \vdash_1^n \Gamma \supset \Lambda$ means that $\Gamma \supset \Lambda$ has a proof of length n so that all cut formulas belong to the collection $\mathcal{U}^\Omega$.

Because of the equality axioms, the $\mathcal{U}^\Omega$ axioms and the rules for Σ reflection and Δ_0^Ω induction on the ordinals it is impossible to prove complete cut elimination for G_Ω. However, the principal formulas of these axioms and rules have rank 0. Therefore, by applying standard techniques of proof theory as presented for example in Girard [7], Schütte [17] or Takeuti [18], one obtains the following weaker result.

Theorem 5 (Cut elimination theorem). *We have for all sequents $\Gamma \supset \Lambda$ and all natural numbers n and r*:

(1) $\quad G_\Omega \vdash_{r+2}^n \Gamma \supset \Lambda \Rightarrow G_\Omega \vdash_{r+1}^{4^n} \Gamma \supset \Lambda,$

(2) $\quad G_\Omega \vdash_{r+1}^n \Gamma \supset \Lambda \Rightarrow G_\Omega \vdash_1^{4_r(n)} \Gamma \supset \Lambda$

where $4_r(n)$ is inductively defined by: $4_0(n) := n$, $4_{r+1}(n) := 4^{4_r(n)}$.

It is an easy exercise to show that the theory PA_Ω^r can be embedded into G_Ω: The number-theoretic axioms, inductive operator axioms, the axiom for the linearity of $<$ on the ordinals and all instances of Δ_0^Ω induction on the natural numbers belong to the $\mathcal{U}^\Omega$ axioms of G_Ω. The Σ reflection axioms of PA_Ω^r are proved in G_Ω by means of the rules for Σ reflection, and the instances of Δ_0^Ω induction on the ordinals of PA_Ω^r can be derived in G_Ω by making use of the corresponding rules for Δ_0^Ω induction on the ordinals. Hence we have the following theorem.

Theorem 6 (Embedding theorem). *If the L_Ω formula φ is provable in PA_Ω^r then there exist natural numbers n and r so that we have*

$$G_\Omega \vdash_r^n \supset \varphi.$$

Combining Theorem 6 and Theorem 5 we obtain the following corollary. It implies in particular that every $\mathcal{U}^\Omega$ formula φ provable in PA_Ω^r has a proof tree in G_Ω which consists of $\mathcal{U}^\Omega$ formulas only.

Corollary 7. *If the L_Ω formula φ is provable in PA_Ω^r then there exists a natural number n so that we have*

$$G_\Omega \vdash_1^n \supset \varphi.$$

In a next step we reduce the $\mathcal{U}^\Omega$ fragment of G_Ω to Peano arithmetic PA. For this purpose we first introduce translations of all L_Ω relation symbols P_A which come with inductive operator forms $A(P, x)$. The following L_1 formulas $I_A^{<n}(s)$ and $I_A^n(s)$ are defined by simultaneous induction on the natural number n:

$$I_A^{<n}(s) := \bigvee_{m<n} I_A^m(s), \qquad I_A^n(s) := A(I_A^{<n}, s).$$

Now let $\varphi[\boldsymbol{\alpha}]$ be an $\mathcal{U}^\Omega$ formula and suppose that n, $\boldsymbol{a}$ are numerals. Then the L_1 formula $\varphi^{(n)}\langle \boldsymbol{a} \rangle$ is inductively defined as follows:

1. If $\varphi[\boldsymbol{\alpha}]$ is an L_1 formula, then $\varphi^{(n)}\langle \boldsymbol{a} \rangle$ is $\varphi[\boldsymbol{\alpha}]$.
2. If $\varphi[\boldsymbol{\alpha}]$ is $(\alpha_i < \alpha_j)$, then $\varphi^{(n)}\langle \boldsymbol{a} \rangle$ is $(a_i <_N a_j)$.
3. If $\varphi[\boldsymbol{\alpha}]$ is $(\alpha_i = \alpha_j)$, then $\varphi^{(n)}\langle \boldsymbol{a} \rangle$ is $(a_i = a_j)$.
4. If $\varphi[\boldsymbol{\alpha}]$ is $P_A^{\alpha_i}(s)$, then $\varphi^{(n)}\langle \boldsymbol{a} \rangle$ is $I_A^{a_i}(s)$.
5. If $\varphi[\boldsymbol{\alpha}]$ is $\neg\psi[\boldsymbol{\alpha}]$, then $\varphi^{(n)}\langle \boldsymbol{a} \rangle$ is $\neg\psi^{(n)}\langle \boldsymbol{a} \rangle$.
6. If $\varphi[\boldsymbol{\alpha}]$ is $(\psi[\boldsymbol{\alpha}]\, j\, \chi[\boldsymbol{\alpha}])$, then $\varphi^{(n)}\langle \boldsymbol{a} \rangle$ is $(\psi^{(n)}\langle \boldsymbol{a} \rangle\, j\, \chi^{(n)}\langle \boldsymbol{a} \rangle)$.
7. If $\varphi[\boldsymbol{\alpha}]$ is $(Qx)\psi[\boldsymbol{\alpha}]$, then $\varphi^{(n)}\langle \boldsymbol{a} \rangle$ is $(Qx)\psi^{(n)}\langle \boldsymbol{a} \rangle$.
8. If $\varphi[\boldsymbol{\alpha}]$ is $(\exists\beta < \alpha_i)\psi[\beta, \boldsymbol{\alpha}]$, then $\varphi^{(n)}\langle \boldsymbol{a} \rangle$ is $\bigvee_{m<a_i} \psi^{(n)}\langle m, \boldsymbol{a} \rangle$.
9. If $\varphi[\boldsymbol{\alpha}]$ is $(\forall\beta < \alpha_i)\psi[\beta, \boldsymbol{\alpha}]$, then $\varphi^{(n)}\langle \boldsymbol{a} \rangle$ is $\bigwedge_{m<a_i} \psi^{(n)}\langle m, \boldsymbol{a} \rangle$.
10. If $\varphi[\boldsymbol{\alpha}]$ is $(\exists\beta)\psi[\beta, \boldsymbol{\alpha}]$, then $\varphi^{(n)}\langle \boldsymbol{a} \rangle$ is $\bigvee_{m<n} \psi^{(n)}\langle m, \boldsymbol{a} \rangle$.
11. If $\varphi[\boldsymbol{\alpha}]$ is $(\forall\beta)\psi[\beta, \boldsymbol{\alpha}]$, then $\varphi^{(n)}\langle \boldsymbol{a} \rangle$ is $\bigwedge_{m<n} \psi^{(n)}\langle m, \boldsymbol{a} \rangle$.

Observe that the length of the formula $\varphi^{(n)}\langle \boldsymbol{a} \rangle$ may depend on every parameter n, $\boldsymbol{a}$. If $\varphi[\boldsymbol{\alpha}]$ is a Δ_0^Ω formula, then the bound n is dispensable, and we write $\varphi\langle \boldsymbol{a} \rangle$ instead of $\varphi^{(n)}\langle \boldsymbol{a} \rangle$. If φ does not contain free ordinal variables, then $\varphi^{(n)}$ stands for $\varphi^{(n)}\langle\ \rangle$.

In the following we collect some properties of this translation of $\mathcal{U}^\Omega$ into L_1. They will be used in the proof of the reduction theorem below.

Lemma 8. *Let $\varphi[\boldsymbol{\alpha}]$ be an $\mathcal{U}^\Omega$ formula and define $\psi[\boldsymbol{\alpha}, \beta] := \varphi^\beta[\boldsymbol{\alpha}]$ where β is a new ordinal variable different from $\boldsymbol{\alpha}$. Then the L_1 formulas $\psi^{(n+1)}\langle \boldsymbol{a}, n \rangle$ and $\varphi^{(n)}\langle \boldsymbol{a} \rangle$ are identical for all natural numbers n, $\boldsymbol{a}$.*

Corollary 9. *Let $\varphi[\boldsymbol{\alpha}]$ be an $\mathcal{U}^\Omega$ formula and define $\psi[\boldsymbol{\alpha}] := (\exists\beta)\varphi^\beta[\boldsymbol{\alpha}]$ where β is a new ordinal variable different from $\boldsymbol{\alpha}$. Then we have for all natural numbers n, $\boldsymbol{a}$:*

$$PA \vdash \varphi^{(n)}\langle \boldsymbol{a} \rangle \to \psi^{(n+1)}\langle \boldsymbol{a} \rangle.$$

Lemma 10. *Let $\varphi[\boldsymbol{\alpha}]$ be a Σ^Ω and $\psi[\boldsymbol{\alpha}]$ a Π^Ω formula and assume that m, n, $\boldsymbol{a}$ are natural numbers so that $m \leq n$. Then we have:*
(1) $PA \vdash \varphi^{(m)}\langle \boldsymbol{a} \rangle \to \varphi^{(n)}\langle \boldsymbol{a} \rangle$,
(2) $PA \vdash \psi^{(n)}\langle \boldsymbol{a} \rangle \to \psi^{(m)}\langle \boldsymbol{a} \rangle$.

Lemma 11. *Let n be a natural number and $(\varphi_i\colon i<n)$ a family of L_1 formulas. Then we have for all natural numbers $m \leq n$:*

$$PA \vdash \bigwedge_{i<n} \left(\left(\bigwedge_{j<i} \varphi_j \right) \to \varphi_i \right) \to \bigwedge_{i<m} \varphi_i.$$

Lemma 8 is proved by induction on the definition of $\varphi[\boldsymbol{\alpha}]$, Lemma 10 by simultaneous induction on the definitions of $\varphi[\boldsymbol{\alpha}]$ and $\psi[\boldsymbol{\alpha}]$. Corollary 9 is an immediate consequence of Lemma 8 and Lemma 11 follows by induction on m.

Finally we extend this translation of $\mathcal{U}^\Omega$ into L_1 to sequents of $\mathcal{U}^\Omega$ formulas. For a finite sequence Γ of $\mathcal{U}^\Omega$ formulas we write Γ_Σ for the set of all Σ^Ω formulas which occur in Γ and Γ_Π for the set of all formulas in Γ which do not belong to Γ_Σ. Hence every formula which occurs in Γ belongs to $\Gamma_\Sigma \cup \Gamma_\Pi$. If $\Gamma[\boldsymbol{\alpha}] \supset \Lambda[\boldsymbol{\alpha}]$ is a sequent of $\mathcal{U}^\Omega$ formulas and if m, n, $\boldsymbol{a}$ are natural numbers, then $(\Gamma \supset \Lambda)^{(m,n)}\langle \boldsymbol{a} \rangle$ is defined to be the L_1 formula

$$\bigvee_{\varphi[\boldsymbol{\alpha}] \in \Gamma_\Sigma} (\neg\varphi)^{(m)}\langle \boldsymbol{a} \rangle \vee \bigvee_{\varphi[\boldsymbol{\alpha}] \in \Gamma_\Pi} (\neg\varphi)^{(n)}\langle \boldsymbol{a} \rangle \vee \bigvee_{\varphi[\boldsymbol{\alpha}] \in \Lambda_\Sigma} \varphi^{(n)}\langle \boldsymbol{a} \rangle \vee \bigvee_{\varphi[\boldsymbol{\alpha}] \in \Lambda_\Pi} \varphi^{(m)}\langle \boldsymbol{a} \rangle.$$

The following reduction theorem provides an interpretation of the $\mathcal{U}^\Omega$ fragment of G_Ω into PA. Its proof is based on an asymmetric treatment of the existential and universal ordinal quantifiers in the sequents $\Gamma \supset \Lambda$.

Theorem 12 (Reduction theorem). *Let $\Gamma[\boldsymbol{\alpha}] \supset \Lambda[\boldsymbol{\alpha}]$ be a sequent of $\mathcal{U}^\Omega$ formulas. Then we have for all natural numbers n and all natural numbers m, $\boldsymbol{a}$ so that $\boldsymbol{a} < m$:*

$$G_\Omega \vdash_1^n \Gamma[\boldsymbol{\alpha}] \supset \Lambda[\boldsymbol{\alpha}] \;\Rightarrow\; PA \vdash (\Gamma \supset \Lambda)^{(m,m+2^n)}\langle \boldsymbol{a} \rangle.$$

Proof. By induction on n. If $\Gamma[\boldsymbol{\alpha}] \supset \Lambda[\boldsymbol{\alpha}]$ is an axiom of G_Ω, then the assertion is trivial; actually, the (translations of) the inductive operator axioms are immediate consequences of the definition of the L_1 formulas $I_A^k(x)$. Otherwise $\Gamma[\boldsymbol{\alpha}] \supset \Lambda[\boldsymbol{\alpha}]$ is the conclusion of a derivation rule. We concentrate on the three critical cases and leave the rest to the reader.

1. $\Gamma[\boldsymbol{\alpha}] \supset \Lambda[\boldsymbol{\alpha}]$ is the conclusion of the rule for Σ reflection

$$\frac{\Gamma[\boldsymbol{\alpha}] \supset \Phi[\boldsymbol{\alpha}],\ \varphi[\boldsymbol{\alpha}]}{\Gamma[\boldsymbol{\alpha}] \supset \Phi[\boldsymbol{\alpha}],\ (\exists\beta)\varphi^\beta[\boldsymbol{\alpha}]}$$

where $\Lambda[\boldsymbol{\alpha}] = \Phi[\boldsymbol{\alpha}],\ (\exists\beta)\varphi^\beta[\boldsymbol{\alpha}]$ and $\varphi[\boldsymbol{\alpha}]$ is a Σ^Ω formula. Then there exists a natural number $k < n$ so that

$$G_\Omega \vdash_1^k \Gamma[\boldsymbol{\alpha}] \supset \Phi[\boldsymbol{\alpha}],\ \varphi[\boldsymbol{\alpha}],$$

and the induction hypothesis implies

$$PA \vdash (\Gamma \supset \Phi)^{(m,m+2^k)}\langle \boldsymbol{a} \rangle \vee \varphi^{(m+2^k)}\langle \boldsymbol{a} \rangle.$$

Hence the assertion follows from Corollary 9 and Lemma 10.

2. $\Gamma[\boldsymbol{\alpha}] \supset \Lambda[\boldsymbol{\alpha}]$ is the conclusion of the rule for Δ_0^Ω induction on the ordinals

$$\frac{\Gamma[\boldsymbol{\alpha}] \supset \Phi[\boldsymbol{\alpha}], (\forall\beta)((\forall\gamma<\beta)\varphi[\gamma, \boldsymbol{\alpha}] \to \varphi[\beta, \boldsymbol{\alpha}])}{\Gamma[\boldsymbol{\alpha}] \supset \Phi[\boldsymbol{\alpha}], (\forall\beta)\varphi[\beta, \boldsymbol{\alpha}]}$$

where $\Lambda[\boldsymbol{\alpha}] = \Phi[\boldsymbol{\alpha}], (\forall\beta)\varphi[\beta, \boldsymbol{\alpha}]$ and $\varphi[\beta, \boldsymbol{\alpha}]$ is a Δ_0^Ω formula. Then there exists a natural number $k < n$ so that

$$G_\Omega \vdash_1^k \Gamma[\boldsymbol{\alpha}] \supset \Phi[\boldsymbol{\alpha}], (\forall\beta)((\forall\gamma<\beta)\varphi[\gamma, \boldsymbol{\alpha}] \to \varphi[\beta, \boldsymbol{\alpha}]),$$

and the induction hypothesis implies

$$PA \vdash (\Gamma \supset \Phi)^{(m,m+2^k)}\langle \boldsymbol{a}\rangle \vee \bigwedge_{i<m}\Big(\bigwedge_{j<i}\varphi\langle j, \boldsymbol{a}\rangle \to \varphi\langle i, \boldsymbol{a}\rangle\Big).$$

Hence the assertion follows from Lemma 11.

3. $\Gamma[\boldsymbol{\alpha}] \supset \Lambda[\boldsymbol{\alpha}]$ is the conclusion of the cut rule

$$\frac{\Gamma[\boldsymbol{\alpha}] \supset \Lambda[\boldsymbol{\alpha}], \varphi[\boldsymbol{\alpha}] \qquad \Gamma[\boldsymbol{\alpha}], \varphi[\boldsymbol{\alpha}] \supset \Lambda[\boldsymbol{\alpha}]}{\Gamma[\boldsymbol{\alpha}] \supset \Lambda[\boldsymbol{\alpha}]}.$$

Then there exist natural numbers k_1, $k_2 < n$ so that

$$G_\Omega \vdash_1^{k_1} \Gamma[\boldsymbol{\alpha}] \supset \Lambda[\boldsymbol{\alpha}], \varphi[\boldsymbol{\alpha}], \tag{1}$$

$$G_\Omega \vdash_1^{k_2} \Gamma[\boldsymbol{\alpha}], \varphi[\boldsymbol{\alpha}] \supset \Lambda[\boldsymbol{\alpha}]. \tag{2}$$

and $rn(\varphi[\boldsymbol{\alpha}]) < 1$. Hence $\varphi[\boldsymbol{\alpha}]$ is an element of $\mathcal{U}^\Omega$, i.e., a Σ^Ω or Π^Ω formula, so that we can apply the induction hypothesis to (1) and (2). By symmetry we may assume without loss of generality that $\varphi[\boldsymbol{\alpha}]$ is a Σ^Ω formula. Then we obtain from (1) by induction hypothesis

$$PA \vdash (\Gamma \supset \Lambda)^{(m,b)}\langle \boldsymbol{a}\rangle \vee \varphi^{(b)}\langle \boldsymbol{a}\rangle \tag{3}$$

for $b := m + 2^{k_1}$. On the other hand, if we replace m by b, the induction hypothesis applied to (2) gives

$$PA \vdash (\Gamma \supset \Lambda)^{(b,c)}\langle \boldsymbol{a}\rangle \vee (\neg\varphi)^{(b)}\langle \boldsymbol{a}\rangle \tag{4}$$

for $c := b + 2^{k_2}$. Since $c = m + 2^{k_1} + 2^{k_2} \leq m + 2^n$, we obtain from (3), (4) and Lemma 10 that

$$PA \vdash (\Gamma \supset \Lambda)^{(m,m+2^n)}\langle \boldsymbol{a}\rangle.$$

This finishes the proof of case 3. In the remaining cases the assertion readily follows from the induction hypothesis. □

Corollary 13. *PA_Ω^r is a conservative extension of PA with respect to all L_1 formulas. Hence we have for all L_1 formulas φ:*

$$PA_\Omega^r \vdash \varphi \Leftrightarrow PA \vdash \varphi.$$

Proof. Obviously PA^r_Ω is an extension of PA. To show the converse direction let φ be an L_1 formula and assume that $PA^r_\Omega \vdash \varphi$. Then Corollary 7 implies the existence of a natural number n so that $G_\Omega \vdash^n_1 \supset \varphi$. By the reduction theorem we can conclude that $PA \vdash \varphi$. $\square$

From Theorem 12 we can also derive that the Σ^Ω fragment of PA^r_Ω can be reduced to PA. The precise formulation is as follows.

Corollary 14 (Σ interpretation of PA^r_Ω). *If φ is a Σ^Ω formula without free ordinal variables and if $PA^r_\Omega \vdash \varphi$, then there exists a natural number n so that $PA \vdash \varphi^{(n)}$.*

Proof. As in the proof before, we apply Corollary 7 to obtain a natural number k so that $G_\Omega \vdash^k_1 \supset \varphi$. Then Theorem 12 implies $PA \vdash \varphi^{(n)}$ for $n := 2^k$. $\square$

Let us briefly summarize what we achieved in this paper: We introduced three natural theories, PA^r_Ω, PA^w_Ω and PA_Ω, for fixed points in arithmetic with ordinals and showed that they are conservative extensions of PA, $\widehat{ID}_1$ and ID_1, respectively. Now it seems intersting to study the proof-theoretic strength of systems between PA^r_Ω and PA_Ω which are obtained by modifications of the principles of induction on the natural numbers and ordinals and to compare these formalisms to subsystems of ID_1 and Kripke–Platek set theory.

References

[1] W. Buchholz, S. Feferman, W. Pohlers and W. Sieg, Iterated Inductive Definitions and Subsystems of Analysis: Recent Proof-Theoretical Studies, Lecture Notes in Math. 897 (Springer, Berlin, 1981).

[2] A. Cantini, Levels of implications and type free theories of partial classifications with approximation operator, Z. Math. Logik Grundlag. Math. 38 (1992).

[3] S. Feferman, Formal theories for transfinite iterations of generalized inductive definitions and some subsystems of analysis, in: A. Kino, J. Myhill and R.E. Vesley, eds., Intuitionism and Proof Theory, Proceedings of the Summer Conference at Buffalo, New York, 1968 (North-Holland, Amsterdam, 1970).

[4] S. Feferman, Generalizing set-theoretical model theory and an analogue theory on admissible sets, in: Hintikka, Niiniluoto, Saarinen, eds., Essays on Mathematical and Philosophical Logic (Reidel, Dordrecht, 1979).

[5] S. Feferman, Iterated fixed-point theories: application to Hancock's conjecture, in: G. Metakides, ed., Patras Logic Symposion (North-Holland, Amsterdam, 1982).

[6] S. Feferman and G. Jäger, Systems of explicit mathematics with non-constructive μ-operator. Part I, submitted for publication, available as preprint, 1991.

[7] J.-Y. Girard, Proof Theory and Logical Complexity, Vol. I (Bibliopolis, Napoli, 1987).

[8] G. Jäger, Beweistheorie von *KPN*, Archiv Math. Logik Grundlagenforschung 20 (1980).

[9] G. Jäger, Zur Beweistheorie der Kripke–Platek-Mengenlehre über den natürlichen Zahlen, Archiv Math. Logik Grundlagenforschung 22 (1982).

[10] G. Jäger, Iterating admissibility in proof theory, in: J. Stern, ed., Proceedings of the Herbrand Symposium, Logic Colloquium '81 (North-Holland, Amsterdam, 1982).

[11] G. Jäger, Theories for Admissible Sets: A Unifying Approach to Proof Theory (Bibliopolis, Napoli, 1986).

[12] Y.N. Moschovakis, Elementary Induction on Abstract Structures (North-Holland, Amsterdam, 1974).

[13] W. Pohlers, Admissibility in proof theory, in: L.J. Cohen, H. Pfeiffer and K.-P. Podewski, eds., Proceedings 6th International Congress Logic, Methodology and Philosophy of Science (North-Holland, Amsterdam, 1982).

[14] W. Pohlers, Contributions of the Schütte school in Munich to proof theory, Appendix in [18].

[15] W. Pohlers, Proof Theory, An Introduction, Lecture Notes in Math. 1407 (Springer, Berlin, 1989).

[16] W. Pohlers, Proof theory and ordinal analysis, Arch. Math. Logic 30 (1991).

[17] K. Schütte, Proof Theory (Springer, Berlin, 1977).

[18] G. Takeuti, Proof Theory, 2nd edition (North-Holland, Amsterdam, 1987).